

# A sharp integral inequality for the dyadic maximal operator and related stability results

Eleftherios N. Nikolidakis

## Abstract

We prove a sharp integral inequality for the dyadic maximal operator due to which the evaluation of the Bellman function of this operator with respect to two variables, is possible. This appears in a more general form in [3]. Our inequality of interest is proved in this article by a simpler and more immediate way. We also study stability results in connection with this inequality, that is we provide a necessary and sufficient condition, for a sequence of functions, under which we obtain equality in the limit.

## 1 Introduction

The dyadic maximal operator on  $\mathbb{R}^n$  is a useful tool in analysis and is defined by

$$\mathcal{M}_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(y)| dy : x \in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}, \quad (1.1)$$

for every  $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ , where the dyadic cubes are those formed by the grids  $2^{-N}\mathbb{Z}^n$ , for  $N = 0, 1, 2, \dots$ . As is well known it satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d \phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_d \phi > \lambda\}} |\phi(y)| dy, \quad (1.2)$$

for every  $\phi \in L^1(\mathbb{R}^n)$  and every  $\lambda > 0$ , from which it is easy to get the following  $L^p$ -inequality

$$\|\mathcal{M}_d \phi\|_p \leq \frac{p}{p-1} \|\phi\|_p, \quad (1.3)$$

for every  $p > 1$  and  $\phi \in L^p(\mathbb{R}^n)$ .

It is easy to see that the weak type inequality (1.2) is best possible. It has also been proved that (1.3) is best possible (see [1], [2] for general martingales and [16] for dyadic ones).

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<sup>0</sup> *E-mail address:* lefteris@math.uoc.gr

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For the study of the dyadic maximal operator it is desirable for one to find refinements of the above mentioned inequalities. Concerning (1.2), improvements have been given in, [9] and [10]. If we consider (1.3), there is a refinement of it if one fixes the  $L^1$ -norm of  $\phi$ . That is we wish to find explicitly the following function (named as Bellman) of two variables  $f$  and  $F$ .

$$B_Q^{(p)}(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}_d \phi)^p : \phi \geq 0, \frac{1}{|Q|} \int_Q \phi = f, \frac{1}{|Q|} \int_Q \phi^p = F \right\}, \quad (1.4)$$

where  $Q$  is a fixed dyadic cube and  $f, F$  are such that  $0 < f^p \leq F$ .

This function was first evaluated in [4]. In fact it has been explicitly computed in a much more general setting of a non-atomic probability space  $(X, \mu)$  equipped with a tree structure  $\mathcal{T}$ , which is similar to the structure of the dyadic subcubes of  $[0, 1]^n$  (see the definition in Section 2). Then we define the associated maximal operator by

$$\mathcal{M}_{\mathcal{T}} \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}, \quad (1.5)$$

for every  $\phi \in L^1(X, \mu)$ .

Moreover (1.2) and (1.3) still hold in this setting and remain sharp. Now if we wish to refine (1.3) we should introduce the so-called Bellman function of the dyadic maximal operator of two variables given by

$$B_{\mathcal{T}}^{(p)}(f, F) = \sup \left\{ \int_X (\mathcal{M}_{\mathcal{T}} \phi)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F \right\}, \quad (1.6)$$

where  $0 < f^p \leq F$ . This function of course generalizes (1.4). In [4] it is proved that

$$B_{\mathcal{T}}^{(p)}(f, F) = F \omega_p \left( \frac{f^p}{F} \right)^p,$$

where  $\omega_p : [0, 1] \rightarrow [1, \frac{p}{p-1}]$ , is defined by  $\omega_p(z) = H_p^{-1}(z)$ , and  $H_p(z)$  is given by  $H_p(z) = -(p-1)z^p + pz^{p-1}$ . As a consequence  $B_{\mathcal{T}}^{(p)}(f, F)$  does not depend on the structure of the tree  $\mathcal{T}$ . The technique for the evaluation of (1.6), that is used in [3], is based on an effective linearization of the dyadic maximal operator that holds on an adequate class of functions called  $\mathcal{T}$ -good (see the definition in Section 2), which is enough to describe the problem that is settled on (1.6). In [7] now a different approach has been given, for the evaluation of (1.6). This was actually done for the Bellman function of three variables in a different way, avoiding the calculus arguments that are given in [4]. More precisely the following is a consequence of the results in [7].

**Theorem A.** *Let  $\phi \in L^p(X, \mu)$  be non-negative, with  $\int_X \phi d\mu = f$ . Then the following inequality is true*

$$\int_X (\mathcal{M}_{\mathcal{T}} \phi)^p d\mu \leq -\frac{1}{p-1} f^p + \frac{p}{p-1} \int_X \phi (\mathcal{M}_{\mathcal{T}} \phi)^{p-1} d\mu. \quad (1.7)$$

This inequality, as one can see in [7] enables us to find a direct proof of the exact evaluation of (1.6). For this evaluation we also need a symmetrization principle that can be found in [7] and which is presented as Theorem 2.1 below, and which is also used in this article for the sharpness of our results. In this paper we will prove the following generalization of Theorem A. By using now the linearization technique that appears in [4], in a more complicated form, we present in Section 3, a proof of the theorem that appears just below (mentioned as Theorem 1), which generalizes Theorem A and which is the following.

**Theorem 1.** *Let  $\phi$  be as in the hypothesis of Theorem A and suppose that  $q \in [1, p]$ . Then the following inequality is true for any  $\beta > 0$ .*

$$\begin{aligned} \int_X (\mathcal{M}_{\mathcal{T}} \phi)^p d\mu \leq & -\frac{q(\beta+1)}{(p-1)q\beta + (p-q)} f^p + \\ & + \frac{p(\beta+1)^q}{(p-1)q\beta + (p-q)} \int_X \phi^q (\mathcal{M}_{\mathcal{T}} \phi)^{p-q} d\mu. \end{aligned} \quad (1.8)$$

Additionally (1.8) is best possible for any given  $q \in [1, p]$ ,  $f > 0$  and  $\beta$  such that  $0 < \beta \leq \frac{1}{p-1}$ . By this we mean that if one fixes the second constant appearing on the right hand side of inequality (1.8), then we cannot increase the absolute value of the first constant appearing in front of  $f^p$ , in a way such that (1.8) still holds.

The following is also true and is an easy consequence of Theorem 1.

**Corollary 1.** *Let  $\phi : (X, \mu) \rightarrow \mathbb{R}^+$  be  $\mathcal{T}$ -good such that  $\int_X \phi d\mu = f$ . Then for every  $q \in [1, p]$  the following inequality holds*

$$\int_X (\mathcal{M}_{\mathcal{T}} \phi)^p d\mu \leq -\frac{q}{p-1} f^p + \left( \frac{p}{p-1} \right)^q \int_X \phi^q (\mathcal{M}_{\mathcal{T}} \phi)^{p-q} d\mu. \quad (1.9)$$

Additionally (1.9) is best possible for any given  $q \in [1, p]$  and  $f > 0$ .

Additionally by using the symmetrization principle that is mentioned below (Theorem 2.1), and Theorem 1 we easily derive inequalities of Hardy type, as described by the following.

**Corollary 2.** *For any  $g : (0, 1] \rightarrow \mathbb{R}^+$  non-increasing such that  $\int_0^1 g(u) du = f$ , the following inequality is true for any  $\beta > 0$  and sharp for any  $\beta$  such that  $0 < \beta \leq \frac{1}{p-1}$ .*

$$\begin{aligned} \int_0^1 \left( \frac{1}{t} \int_0^t g(u) du \right)^p dt \leq & -\frac{q(\beta+1)}{(p-1)q\beta + (p-q)} f^p + \\ & + \frac{p(\beta+1)^q}{(p-1)q\beta + (p-q)} \int_0^1 \left( \frac{1}{t} \int_0^t g(u) du \right)^{p-q} g^q(t) dt. \end{aligned} \quad (1.10)$$

For the case  $q = 1$ , and the value  $\beta = \frac{1}{p-1}$ , inequality (1.10) is well known and is in fact equality, as can be seen by applying a simple integration by parts argument. Note also that these types of inequalities involve parameters inside them, and the validity of them still remains true as much as their sharpness. These type of inequalities as (1.8) or (1.10), generalize inequality (1.7) in two important directions, and this is the appearance of the two parameters involved. We need also to add that inequality (1.8), seems to be powerfull (not only sharp), for any  $q \in [1, p]$ , and this is due to the fact that for any such  $q$ , as one can see in [3], it can be also used for the evaluation of the Bellman function (1.6).

We also note that inequality (1.8) is also a consequence of the results in [3], where it is proved a more general inequality which involves also the parameter  $A = \int_X \phi^q d\mu$ . In this paper we ignore this parameter and give a more direct proof of (1.8). We also give the sharpness of this inequality by different methods that appear in [3].

Additionally in our last section we provide a result which, under the minimum of hypothesis, describes us when we do have equality in (1.8). More precisely we prove the following.

**Theorem 2.** *Let  $(\phi_n)_n$  be a sequence of functions as in the hypothesis of Theorem A, satisfying also  $\int_X \phi_n^p d\mu = F$ , for every  $n \in N$ . Consider also the quantity  $A_0(\beta) = \frac{(q-1)\beta}{(\beta+1)^q} + \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}}$ , where  $\beta \in [0, \frac{1}{p-1})$ , and  $1 < q < p$ . Suppose also that the variables  $\beta, f, F, q$  and  $p$  are connected by the relation*

$$F(\beta+1)^{p-q} = A_0(\beta)F(\beta+1)^p + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p \quad (1.11)$$

*Then  $(\phi_n)_n$  satisfies equality in the limit in (1.8), if and only if the following is true*

$$\lim_n \int_X |\mathcal{M}_{\mathcal{T}} \phi_n - (\beta+1)\phi_n|^p d\mu = 0, \quad (1.12)$$

*which means exactly that  $(\phi_n)_n$  behaves approximately like an eigenfunction sequence for the eigenvalue  $\beta+1$ .*

At last we mention that the evaluation of (1.6) has been given by an alternative method in [11], while certain Bellman functions corresponding to several problems in harmonic analysis, have been studied in [5], [6], [12], [13], [14] and [15].

## 2 Preliminaries

Let  $(X, \mu)$  be a non-atomic probability space. We give the following from [4] or [7].

**Definition 2.1.** *A set  $\mathcal{T}$  of measurable subsets of  $X$  will be called a tree if the following are satisfied*

- i)  $X \in \mathcal{T}$  and for every  $I \in \mathcal{T}$ ,  $\mu(I) > 0$ .*

- ii) For every  $I \in \mathcal{T}$  there corresponds a finite or countable subset  $C(I)$  of  $\mathcal{T}$  containing at least two elements such that
- a) the elements of  $C(I)$  are pairwise disjoint subsets of  $I$
  - b)  $I = \bigcup C(I)$ .

iii)  $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ , where  $\mathcal{T}_{(0)} = \{X\}$  and

$$\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} C(I).$$

iv) The following holds

$$\lim_{m \rightarrow \infty} \sup_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$$

For the proof of Theorem 1 we will need an effective linearization for the operator  $\mathcal{M}_{\mathcal{T}}$  that was introduced in [4]. We describe it as appears there and use it in the sequel.

For every  $\phi \in L^1(X, \mu)$ , non negative, and  $I \in \mathcal{T}$  we define  $\text{Av}_I(\phi) = \frac{1}{\mu(I)} \int_I \phi d\mu$ . We will say that  $\phi$  is  $\mathcal{T}$ -good if the set

$$\mathcal{A}_{\phi} = \{x \in X : \mathcal{M}_{\mathcal{T}} \phi(x) > \text{Av}_I(\phi) \text{ for all } I \in \mathcal{T} \text{ such that } x \in I\}$$

has  $\mu$ -measure zero.

Let now  $\phi$  be  $\mathcal{T}$ -good and  $x \in X \setminus \mathcal{A}_{\phi}$ . We define  $I_{\phi}(x)$  to be the largest in the nonempty set

$$\{I \in \mathcal{T} : x \in I \text{ and } \mathcal{M}_{\mathcal{T}} \phi(x) = \text{Av}_I(\phi)\}.$$

Now given  $I \in \mathcal{T}$  let

$$\begin{aligned} A(\phi, I) &= \{x \in X \setminus \mathcal{A}_{\phi} : I_{\phi}(x) = I\} \subseteq I \text{ and} \\ S_{\phi} &= \{I \in \mathcal{T} : \mu(A(\phi, I)) > 0\} \cup \{X\}. \end{aligned}$$

Obviously then

$$\mathcal{M}_{\mathcal{T}} \phi = \sum_{I \in S_{\phi}} \text{Av}_I(\phi) \chi_{A(\phi, I)}, \quad \mu\text{-a.e.},$$

where  $\chi_E$  is the characteristic function of  $E$ . We also define the following correspondence  $I \rightarrow I^*$  by:  $I^*$  is the smallest element of  $\{J \in S_{\phi} : I \subsetneq J\}$ . It is defined for every  $I \in S_{\phi}$ , except  $X$ . Also it is obvious that the  $A(\phi, I)$ 's are pairwise disjoint and that

$$\mu \left( \bigcup_{I \notin S_{\phi}} A(\phi, I) \right) = 0,$$

so that

$$\bigcup_{I \in S_{\phi}} A(\phi, I) \approx X,$$

where by  $A \approx B$  we mean that

$$\mu(A \setminus B) = \mu(B \setminus A) = 0.$$

Now the following is true (see [4]).

**Lemma 2.1.** *Let  $\phi$  be  $\mathcal{T}$ -good*

- i) *If  $I, J \in S_\phi$  then either  $A(\phi, J) \cap I = \emptyset$  or  $J \subseteq I$ .*
- ii) *If  $I \in S_\phi$  then there exists  $J \in C(I)$  such that  $J \notin S_\phi$ .*
- iii) *For every  $I \in S_\phi$  we have that  $I \approx \bigcup_{\substack{J \in S_\phi \\ J \subseteq I}} A(\phi, J)$ .*
- iv) *For every  $I \in S_\phi$  we have that*

$$A(\phi, I) = I \setminus \bigcup_{\substack{J \in S_\phi \\ J^* = I}} J,$$

*so that*

$$\mu(A(\phi, I)) = \mu(I) - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J).$$

From the above we see that

$$\text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{\substack{J \in S_\phi \\ J \subseteq I}} \int_{A(\phi, J)} \phi \, d\mu.$$

In the sequel we will also need the notion of the decreasing rearrangement of a  $\mu$ -measurable function defined on  $X$ . This is given by the following equation

$$\phi^*(t) = \sup_{\substack{e \subseteq X \\ \mu(e) \geq t}} \left[ \inf_{x \in e} |\phi(x)| \right], \quad t \in (0, 1].$$

This is a non-increasing, left continuous function defined on  $(0, 1]$  and equimeasurable to  $|\phi|$  (that is  $\mu(\{|\phi| > \lambda\}) = |\{\phi^* > \lambda\}|$ , for any  $\lambda > 0$ ). A more intuitive definition of  $\phi^*$  is that it describes a rearrangement of the values of  $|\phi|$  in decreasing order. We are now ready to state the following, which appears in [7] and can be viewed as a symmetrization principle for the dyadic maximal operator.

**Theorem 2.1.** *The following equality is true*

$$\begin{aligned} & \sup \left\{ \int_K G_1(\mathcal{M}_\mathcal{T} \phi) G_2(\phi) \, d\mu : \phi^* = g, \phi \geq 0, \right. \\ & \quad \left. K \text{ measurable subset of } X \text{ with } \mu(K) = k \right\} = \\ & \quad = \int_0^k G_1\left(\frac{1}{t} \int_0^t g\right) G_2(g(t)) \, dt, \quad (2.1) \end{aligned}$$

where  $G_i : [0, +\infty) \rightarrow [0, +\infty)$  are increasing functions for  $i = 1, 2$ , while  $g : (0, 1] \rightarrow \mathbb{R}^+$  is non-increasing. Additionally the supremum in (2.1) is attained by some  $(\phi_n)$  such that  $\phi_n^* = g$ , for every pair of functions  $(G_1, G_2)$ .

### 3 Proof of the inequality (1.8)

We now proceed to the

*Proof of Theorem 1.*

Let  $\phi : (X, \mu) \rightarrow \mathbb{R}^+$  be  $\mathcal{T}$ -good such that  $\int_X \phi d\mu = f$  and let  $q \in (1, p]$ . (The case  $q = 1$  can be handled easily, if we consider a sequence  $(q_n)_n$ , of elements of  $(1, p]$ , tending to  $q = 1$ , and applying the result for every  $q_n$ ). We consider the quantity

$$k_q = \int_X \phi^q (\mathcal{M}_{\mathcal{T}} \phi)^{p-q} d\mu.$$

By the definition of the linearization of the dyadic maximal operator we have that

$$k_q = \sum_{I \in S_\phi} \int_{A(\phi, I)} \phi^q d\mu \cdot y_I^{p-q}. \quad (3.1)$$

By Hölder's inequality now, since  $q > 1$ , we have that

$$\int_{A(\phi, I)} \phi^q d\mu \geq \frac{1}{\alpha_I^{q-1}} \left( \int_{A(\phi, I)} \phi d\mu \right)^q, \quad (3.2)$$

where  $A(\phi, I) = I \setminus \bigcup_{J \in S_\phi, J^* = I} J$ , in view of Lemma 2.1 iv), and so  $\alpha_I = \mu(A(\phi, I)) = \mu(I) - \sum_{J \in S_\phi, J^* = I} \mu(J)$ . Thus (3.1) in view of (3.2) becomes

$$\begin{aligned} k_q &\geq \sum_{I \in S_\phi} y_I^{p-q} \frac{\left( \int_I \phi d\mu - \sum_{J \in S_\phi, J^* = I} \int_J \phi d\mu \right)^q}{\left( \mu(I) - \sum_{J \in S_\phi, J^* = I} \mu(J) \right)^{q-1}} = \\ &= \sum_{I \in S_\phi} y_I^{p-q} \frac{\left( \mu(I) y_I - \sum_{J \in S_\phi, J^* = I} \mu(J) y_J \right)^q}{\left( \mu(I) - \sum_{J \in S_\phi, J^* = I} \mu(J) \right)^{q-1}}. \end{aligned} \quad (3.3)$$

We use now Hölder's inequality in the following form

$$\frac{(\lambda_1 + \lambda_2 + \dots + \lambda_m)^q}{(\sigma_1 + \sigma_2 + \dots + \sigma_m)^{q-1}} \leq \frac{\lambda_1^q}{\sigma_1^{q-1}} + \frac{\lambda_2^q}{\sigma_2^{q-1}} + \dots + \frac{\lambda_m^q}{\sigma_m^{q-1}}, \quad (3.4)$$

which holds for every  $\lambda_i \geq 0$ ,  $\sigma_i > 0$ , since  $q > 1$ .

We consider now an arbitrary  $\beta$ , such that  $0 \leq \beta \leq \frac{1}{p-1}$ . We set for any  $I \in S_\phi$

$$\tau_I = (\beta + 1) - \beta \rho_I, \quad \text{where} \quad \rho_I = \frac{\mu(A(\phi, I))}{\mu(I)} = \frac{\alpha_I}{\mu(I)},$$

thus concluding that  $\tau_I > 0$ . For this choice of  $\tau_I$ , we have that

$$\tau_I \mu(I) - (\beta + 1) \sum_{J \in S_\phi, J^* = I} \mu(J) = \mu(I) - \sum_{J \in S_\phi, J^* = I} \mu(J). \quad (3.5)$$

Thus using (3.4) and (3.5), we have from (3.3) that

$$\begin{aligned} k_q &\geq \sum_{I \in S_\phi} y_I^{p-q} \left\{ \frac{(\mu(I) y_I)^q}{(\mu(I) \tau_I)^{q-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \frac{(\mu(J) y_J)^q}{((\beta + 1) \mu(J))^{q-1}} \right\} = \\ &= \sum_{I \in S_\phi} \mu(I) \frac{y_I^p}{\tau_I^{q-1}} - \sum_{I \in S_\phi} y_I^{p-q} \sum_{\substack{J \in S_\phi \\ J^* = I}} \frac{y_J^q}{(\beta + 1)^{q-1}} \mu(J). \end{aligned} \quad (3.6)$$

By the definitions now of  $S_\phi$ , and the correspondence  $I \rightarrow I^*$  for  $I \neq X$ , we conclude from (3.6) that

$$\begin{aligned} k_q &\geq \sum_{I \in S_\phi} \mu(I) \frac{y_I^p}{\tau_I^{q-1}} - \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{1}{(\beta + 1)^{q-1}} y_I^q (y_{I^*})^{p-q} \mu(I) = \\ &= \sum_{I \in S_\phi} \frac{1}{\rho_I} \alpha_I \frac{y_I^p}{((\beta + 1) - \beta \rho_I)^{q-1}} - \frac{1}{p} \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{p y_I^q (y_{I^*})^{p-q}}{(\beta + 1)^{q-1}} \mu(I). \end{aligned} \quad (3.7)$$

We now use the following elementary inequality,

$$p x^q \cdot y^{p-q} \leq q x^p + (p - q) y^p,$$

which holds since  $1 < q \leq p$ , for any  $x, y > 0$ . By (3.7) we thus have

$$\begin{aligned} k_q &\geq \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{y_I^p}{((\beta + 1) - \beta \rho_I)^{q-1}} - \frac{1}{p} \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{[q y_I^p + (p - q) (y_{I^*})^p]}{(\beta + 1)^{q-1}} \mu(I) = \\ &= \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{y_I^p}{((\beta + 1) - \beta \rho_I)^{q-1}} - \frac{p - q}{p} \frac{1}{(\beta + 1)^{q-1}} \sum_{\substack{I \in S_\phi \\ I \neq X}} (y_{I^*})^p \mu(I) - \\ &\quad - \frac{q}{p} \frac{1}{(\beta + 1)^{q-1}} \sum_{I \in S_\phi} y_I^p \mu(I) + \frac{q}{p} \frac{1}{(\beta + 1)^{q-1}} y_X^p. \end{aligned} \quad (3.8)$$

By using now Lemma 2.1 iv), and the definition of the correspondence  $I \rightarrow I^*$ , we have that

$$\sum_{\substack{I \in S_\phi \\ I \neq X}} (y_{I^*})^p \mu(I) = \sum_{I \in S_\phi} y_I^p (\mu(I) - \alpha_I),$$



thus (3.8) gives

$$k_q \geq \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{1}{((\beta+1) - \beta\rho_I)^{q-1}} y_I^p - \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}} \sum_{I \in S_\phi} (\mu(I) - \alpha_I) y_I^p - \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} \sum_{I \in S_\phi} \mu(I) y_I^p + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} y_X^p.$$

After some simple cancellations we conclude that

$$k_q \geq \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \left( \frac{1}{((\beta+1) - \beta\rho_I)^{q-1}} - \frac{1}{(\beta+1)^{q-1}} \right) y_I^p + \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}} \sum_{I \in S_\phi} \alpha_I y_I^p + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} y_X^p. \quad (3.9)$$

Note now that

$$\frac{1}{((\beta+1) - \beta x)^{q-1}} - \frac{1}{(\beta+1)^{q-1}} \geq \frac{(q-1)\beta x}{(\beta+1)^q},$$

by the mean value theorem on derivatives for all  $x \in [0, 1]$ , so by (3.9) we have as a consequence that

$$\begin{aligned} k_q &\geq \sum_{I \in S_\phi} \left[ \frac{\alpha_I (q-1)\beta\rho_I}{\rho_I (\beta+1)^q} \right] y_I^p + \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}} \sum_{I \in S_\phi} \alpha_I y_I^p + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} y_X^p \\ &= \sum_{I \in S_\phi} \left[ \frac{(q-1)\beta}{(\beta+1)^q} + \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}} \right] \alpha_I y_I^p + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p, \end{aligned} \quad (3.10)$$

and we have derived inequality (1.9)

At this point we give the following.

*Proof of Corollary 2.*

Let  $g : (0, 1] \rightarrow \mathbb{R}^+$  be non-increasing, such that  $\int_0^1 g(u) du = f$ . Fix a nonatomic probability space  $(X, \mu)$  equipped with a tree structure  $\mathcal{T}$ , for which the  $\mathcal{T}$ -step functions (which are included in the  $\mathcal{T}$ -good functions) are dense in  $L^p(X, \mu)$ . Then (1.9) is true for every  $L^p$ -function  $\phi$ , as can be easily seen by standard approximations arguments. Applying Theorem 2.1 for the pair of functions

$$(G_1(t) = t^p, G_2(t) = 1) \quad \text{and} \quad (G'_1(t) = t^{p-q}, G'_2(t) = t^q)$$

we conclude that there exists  $\phi_n : (X, \mu) \rightarrow \mathbb{R}^+$  such that  $\phi_n^* = g$ , for which

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^p dt, \quad (3.11)$$

and

$$\lim_n \int_X \phi_n^q (\mathcal{M}_{\mathcal{T}} \phi_n)^{p-q} d\mu = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt. \quad (3.12)$$

Applying (1.8) for every  $(\phi_n)$ , and taking the limits as  $n \rightarrow \infty$ , we conclude by (3.11) and (3.12) the statement of Corollary 2.

We now prove that (1.10) is best possible. We proceed to this as follows: We first treat the case where  $\beta = \frac{1}{p-1}$ . We consider the following continuous, decreasing function  $g_\alpha(t) = ct^{-\alpha}$ , defined in  $(0, 1]$ , where  $c = f(1 - \alpha)$ , and  $\alpha \in (0, \frac{1}{p})$ . Then it is easy to show that  $\int_0^1 g_\alpha(u) du = f$ , while  $g_\alpha \in L^p(0, 1)$ .

Note that for any  $t \in (0, 1]$  the following equality holds  $\frac{1}{t} \int_0^t g(u) du = (\frac{p}{p-1})g(t)$ . So we consider the difference

$$J = \int_0^1 \left( \frac{1}{t} \int_0^t g_\alpha \right)^p dt - \left( \frac{p}{p-1} \right)^q \int_0^1 g_\alpha^q(t) \left( \frac{1}{t} \int_0^t g_\alpha \right)^{p-q} dt$$

which is equal to

$$J = \left( \frac{1}{1-\alpha} \right)^p \int_0^1 g_\alpha^p(t) dt - \left( \frac{p}{p-1} \right)^q \left( \frac{1}{1-\alpha} \right)^{p-q} \int_0^1 g_\alpha^p(t) dt.$$

Since  $\int_0^1 g_\alpha^p(t) dt = f^p(1-\alpha)^p \frac{1}{1-\alpha p}$ , we have by the above evaluation of  $J$ , that

$$\begin{aligned} J &= \frac{f^p}{1-\alpha p} - \left( \frac{p}{p-1} \right)^q (1-\alpha)^q \frac{f^p}{1-\alpha p} = \\ &= -\frac{f^p}{1-\alpha p} \left[ \left( \frac{p}{p-1} \right)^q (1-\alpha)^q - 1 \right] = -f^p G(\alpha), \end{aligned}$$

where  $G(\alpha)$  is defined for any  $\alpha \in (0, \frac{1}{p})$  by  $G(\alpha) = \frac{(\frac{p}{p-1})^q (1-\alpha)^q - 1}{1-\alpha p}$ . But as it is easily seen, by using de L'Hospital's rule,

$$\lim_{\alpha \rightarrow 1/p^-} G(\alpha) = -q \left( 1 - \frac{1}{p} \right)^{q-1} \left( \frac{p}{p-1} \right)^q \left( -\frac{1}{p} \right) = \frac{q}{p-1}.$$

We now prove the sharpness of (1.10), for any  $\beta$  such that  $0 < \beta < \frac{1}{p-1}$ .

We fix such a  $\beta$ , and we consider the following continuous, decreasing function  $g_\beta(t) = ct^{-\alpha}$ , defined in  $(0, 1]$ , where  $c = f(1 - \alpha)$ , and  $\alpha = \frac{\beta}{\beta+1}$ . Then  $\alpha \in (0, \frac{1}{p})$ , and it is easy to see that  $\int_0^1 g_\beta(u) du = f$ , while for any  $\beta$  as above,  $g_\beta \in L^p(0, 1)$ .

Moreover  $\int_0^1 g_\beta^p(u) du = \frac{f^p}{(\beta+1)^p} \frac{\beta+1}{1-\beta(p-1)}$ . Note that for any  $t \in (0, 1]$  the following equality holds  $\frac{1}{t} \int_0^t g_\beta(u) du = (\beta+1)g_\beta(t)$ . We then consider the difference

$$\begin{aligned} J &= \int_0^1 g_\beta^q(t) \left( \frac{1}{t} \int_0^t g_\beta \right)^{p-q} dt - \\ &\quad - \left[ \frac{(q-1)\beta}{(\beta+1)^q} + \frac{p-q}{p} \left( \frac{1}{\beta+1} \right)^{q-1} \right] \int_0^1 \left( \frac{1}{t} \int_0^t g_\beta \right)^p dt \quad (3.13) \end{aligned}$$

Then due to the above mentioned relations, we can see easily after some simple calculations that  $J = \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p$ . The proof of Corollary 2 is now complete.  $\square$

Now for the proof of Theorem 2, we need to prove the sharpness of (1.8). This is easy now to show, since by Theorem 2.1 for any  $g : (0, 1] \rightarrow \mathbb{R}^+$  non increasing, there exists a sequence  $\phi_n : (X, \mu) \rightarrow \mathbb{R}^+$  of rearrangements of  $g$  such that

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu = \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^p dt \quad (3.14)$$

and

$$\lim_n \int_X \phi_n^q (\mathcal{M}_{\mathcal{T}} \phi_n)^{p-q} d\mu = \int_0^1 g^q(t) \left( \frac{1}{t} \int_0^t g \right)^{p-q} dt. \quad (3.15)$$

We discuss now the case where  $0 < \beta < \frac{1}{p-1}$ , and we consider the function  $g_\beta$  (denoted now as  $g$ ), constructed in the proof of Corollary 1. We choose, for every  $n \in N$ , a rearrangement  $\phi_n$  of  $g$  such that

$$\left| \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^p dt - \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \right| \leq \frac{1}{n}$$

and

$$\left| \int_0^1 g^q(t) \left( \frac{1}{t} \int_0^t g \right)^{p-q} dt - \int_X \phi_n^q (\mathcal{M}_{\mathcal{T}} \phi_n)^{p-q} d\mu \right| \leq \frac{1}{n}$$

Then, by the choice of  $g$ , we conclude that (1.9) is best possible. The case  $\beta = \frac{1}{p-1}$  is entirely similar, so we omit it. The proof of Theorem 1, is now complete.  $\square$

## 4 Proof of Theorem 2

*Proof.* We begin by stating inequality (1.8), in the following equivalent form, which is exactly (3.10)

$$\int_X (\mathcal{M}_{\mathcal{T}} \phi)^{p-q} \phi^q d\mu \geq A_0(\beta) \int_X (\mathcal{M}_{\mathcal{T}} \phi)^p d\mu + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p. \quad (4.1)$$

Here  $A_0(\beta)$  denotes the quantity defined in the statement of Theorem 2. For the one direction of the proof we suppose that  $(\phi_n)_n$  satisfies  $\int_X \phi_n^p d\mu = F$ , for every  $n \in N$ , the relation stated in (1.12), as much as equality in the limit in (4.1). It is immediate then that equality (1.11) is true.

For the opposite direction we suppose that we are given  $f, F$  such that  $0 < f^p \leq F$  and  $q, p$  for which  $1 < q < p$ . We begin this proof by showing that under the above conditions, there exists unique  $\beta \in [0, \frac{1}{p-1})$ , which satisfies (1.11). We proceed for this as follows. We consider the function  $G$  defined as follows.

$$G(\beta) = \frac{1}{(\beta+1)^{p-1} [1 - \beta(p-1)]}, \quad (4.2)$$

where  $\beta \in [0, \frac{1}{p-1})$ . Then  $\frac{1}{G(\beta)} = H_p(\beta + 1)$ , where  $H_p$  is as defined in the Introduction. Since  $G(0) = 1$  and  $G(\frac{1}{p-1}^-) = +\infty$ , we conclude, because of the monotonicity of  $G$ , that there exists unique  $\beta$  in the above range, such that  $G(\beta) = \frac{F}{f^p}$ . This implies as one can easily see, by using only the definition of the function  $G$ , and the particular choice of the variable  $\beta$ , that

$$F(\beta + 1)^{p-q} - A_0(\beta)F(\beta + 1)^p = \frac{q}{p} \frac{1}{(\beta + 1)^{q-1}} f^p.$$

Moreover this value of  $\beta$  satisfies obviously  $H_p(\beta + 1) = \frac{f^p}{F}$ , which is equivalent to  $\beta + 1 = \omega_p(\frac{f^p}{F})$ , and conversely as one can easily see that the solution of (1.11), is given by this last mentioned formula. At this point we give the following.

*Remark. 4.1 For the proof of Theorem 2 we consider a sequence of non negative functions with fixed  $L^p$  and  $L^1$  norms, which gives equality in the limit in (4.1) for a certain value of the variable  $\beta$ , which depends on these norms. That is we insert the  $L^p$  norm as a variable and we fix the variable  $\beta$ . We need to do this and this is more clear by the proof of the sharpness of Theorem 2, where we constructed the particular non increasing function  $g_\beta$ , in view of the symmetrization principle mentioned in the Preliminaries Section (see Theorem 2.1).*

We get back to the proof of Theorem 2, so we suppose that we are given a sequence of non negative functions in  $L^p$ ,  $(\phi_n)_n$  whose elements satisfy  $\int_X \phi_n = f \, d\mu = f$  and  $\int_X \phi_n^p \, d\mu = F$ . Then for the  $q$ -integrals of its elements, if we suppose that they satisfy  $\int_X \phi_n^q \, d\mu = A_n$  for every  $n \in N$ , we may, by passing to a subsequence of  $(A_n)_n$ , it is satisfied that this last sequence tends to a fixed constant which we call  $A$ . This is because, by the above definition of the respective variables, the inequality  $f^q \leq A_n \leq F^{q/p}$ , that is the sequence  $(A_n)_n$  is bounded, and as we can easily see by the proof that comes afterwards we may suppose that  $(A_n)_n$  is constant, that is  $\int_X \phi_n^q \, d\mu = A$ , for every  $n \in N$ .

We also assume that satisfies equality in the limit in (4.1), for the choice of  $\beta$ , which is described above. We now go back to the proof of Theorem 1, and examine where inequalities were used. In these inequalities now we have equality in the limit for our sequence 2. The first one that is used is the following

$$\int_{A(\phi, I)} \phi^q \, d\mu \geq \frac{1}{\alpha^{q-1}} \left( \int_{A(\phi, I)} \phi \, d\mu \right)^q,$$

where  $\alpha = \mu(A(\phi, I))$ , which is exactly inequality (3.2). Additionally the right member of this inequality equals

$$\frac{\left( \int_I \phi \, d\mu - \sum_{J \in S_\phi, J^* = I} \int_J \phi \, d\mu \right)^q}{\left( \mu(I) - \sum_{J \in S_\phi, J^* = I} \mu(J) \right)^{q-1}},$$

which in turn is greater or equal than

$$\mu(I) \frac{y_I^q}{(\tau_I)^{q-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}},$$

where  $\tau_I = (\beta + 1) - \beta \rho_I$ . Since now we have equality in the limit in (4.1), we conclude that in the inequality

$$0 \leq \sum_{I \in S_\phi} y_I^{p-q} \left\{ \int_{A(\phi, I)} \phi^q d\mu - \left[ \mu(I) \frac{y_I^q}{(\tau_I)^{q-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}} \right] \right\} \quad (4.3)$$

we have equality in the limit as  $\phi$  moves along  $(\phi_n)_n$ , and  $S_\phi$  is replaced by  $S_{\phi_n}$ . That is the right member of (4.3), tends to zero for our sequence  $(\phi_n)_n$ .

Additionally every term on the sum in (4.3), is non negative by the comments mentioned right above. Thus since  $y_I \geq f$ , for every  $I \in S_\phi$ , we have that also the following sum tends to zero,

$$0 \leq \sum_{I \in S_\phi} f^{p-q} \left\{ \int_{A(\phi, I)} \phi^q d\mu - \left[ \mu(I) \frac{y_I^q}{(\tau_I)^{q-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}} \right] \right\}, \quad (4.4)$$

as  $\phi$  is moving along  $(\phi_n)_n$ . Cancelling then the term  $f^{p-q}$ , and using Lemma 2.1 iii) and the integral assumptions for every  $\phi \in (\phi_n)_n$  we immediately conclude that the following inequality is true

$$A \geq \sum_{I \in S_\phi} \left[ \mu(I) \frac{y_I^q}{(\tau_I)^{q-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}} \right], \quad (4.5)$$

and is also equality in the limit for our sequence  $(\phi_n)_n$ . We substitute now in  $\tau_I$  its value according to its definition and we get the inequality

$$A \geq \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{y_I^q}{[(\beta + 1) - \beta \rho_I]^{q-1}} - \sum_{I \in S_\phi} \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}}, \quad (4.6)$$

with equality in the limit. Now the left hand side of this inequality equals

$$\sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{y_I^q}{[(\beta + 1) - \beta \rho_I]^{q-1}} - \sum_{I \in S_\phi, I \neq X} \mu(J) \frac{y_J^q}{(\beta + 1)^{q-1}},$$

which in turn equals to

$$\sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \left\{ \frac{1}{[(\beta + 1) - \beta \rho_I]^{q-1}} - \frac{1}{(\beta + 1)^{q-1}} \right\} y_I^q + \frac{y_X^q}{(\beta + 1)^{q-1}}.$$

But in the proof of Theorem 1, we have used the inequality

$$\frac{1}{((\beta + 1) - \beta \rho_I)^{q-1}} - \frac{1}{(\beta + 1)^{q-1}} \geq \frac{(q-1)\beta \rho_I}{(\beta + 1)^q},$$

for every  $I \in S_\phi$ , and by the same arguments that were used above, by replacing  $y_I^p$  with  $y_I^q$ , we conclude that we should have equality in the limit in the following inequality

$$A \geq \sum_{I \in S_\phi} \frac{\alpha_I}{\rho_I} \frac{(q-1)\beta \rho_I}{(\beta + 1)^q} y_I^q + \frac{f^q}{(\beta + 1)^{q-1}} = \frac{(q-1)\beta}{(\beta + 1)^q} \int_X (\mathcal{M}_T \phi)^q d\mu + \frac{f^q}{(\beta + 1)^{q-1}}. \quad (4.7)$$

This equality gives us the following one

$$\lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = (1 + \frac{1}{\beta}) \frac{(\beta + 1)^{q-1} A - f^q}{q-1} \quad (4.8)$$

for our certain value of  $\beta$ , that satisfies  $\beta + 1 = \omega_p(\frac{f^p}{F})$ , as mentioned above.

But the right side of (4.8), is minimized exactly when  $\beta + 1 = \omega_q(\frac{f^q}{A})$ , as is mentioned in [4], where in our case we replace  $p$  by  $q$ , and  $F$  by  $A$ . From the above we conclude that the value of  $A$  is such that  $\beta + 1 = \omega_q(\frac{f^q}{A}) = \omega_p(\frac{f^p}{F})$ , and by the proof of the related theorem, concerning the Bellman function of the dyadic maximal operator as presented in [4] (page 326), we obtain as a consequence that the following equality is true

$$\lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = \omega_q(\frac{f^q}{A})^q A, \quad (4.9)$$

that is,  $(\phi_n)_n$  behaves as an extremal sequence for the Bellman function  $B_T^{(q)}(f, A)$ . In [8] now it is proved that all such sequences behave like  $L^q$  approximate eigenfunctions for the eigenvalue  $\omega_q(\frac{f^q}{A})$ , which equals  $\beta + 1$ . That is the following holds

$$\lim_n \int_X |(\mathcal{M}_T \phi_n) - (\beta + 1)\phi_n|^q d\mu = 0. \quad (4.10)$$

Our purpose was to show the same equality, but with  $p$  in place of  $q$ . This is now not difficult to show, because of the following argument. Since (4.10) is true and by a well known theorem in measure theory, we conclude that there exists a subsequence of  $(\phi_n)_n$  (without loss of generality we call it again  $(\phi_n)_n$ ) for which  $(\mathcal{M}_T \phi_n) - (\beta + 1)\phi_n \rightarrow 0$ , uniformly  $\mu$ -almost everywhere, that is there exists a decreasing sequence  $(A_n)_n$  of  $\mu$ -measurable subsets of  $X$ , for which the following hold  $\mu(A_n) \rightarrow 0$ , and

$$|(\mathcal{M}_T \phi_n)(x) - (\beta + 1)\phi_n(x)| \leq \frac{1}{n}, \quad (4.11)$$

for every  $x \in X \setminus A_n$ , and for every  $n \in N$ . Define now  $h_n(x) = (\mathcal{M}_{\mathcal{T}} \phi_n)(x) - (\beta + 1)\phi_n(x)$ , for every  $x \in X$ . Then the following inequality is true for every  $n \in N$ .

$$\int_X |h_n|^p d\mu = \int_{X \setminus A_n} |h_n|^q |h_n|^{p-q} d\mu + \int_{A_n} |h_n|^p d\mu.$$

The first integral of the right side of this last equation is less or equal than  $\frac{1}{n^{p-q}} \int_{X \setminus A_n} |h_n|^q d\mu$ , which tends to zero in view of (4.10). The second also tends to zero in view of the following reasons. First of all it is less or equal than  $(\beta + 2)^p \int_{A_n} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu$ , because the tree  $\mathcal{T}$ , differentiates  $L^1(X, \mu)$ , by Definition 2.1. Secondly this last quantity tends to zero because of Theorem 5 in [4], if we let  $k$  to tend to zero. That is the Bellman function of three variables  $f, F$  and  $k$ , as described in that paper tends to zero as  $k$  does, and this comes easily by the expression (7.14) that appears there.

□

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Nikolidakis Eleftherios, Visiting Professor, University of Ioannina, Department of Mathematics, GR 45110, Panepistimioupolis, Greece.